

Inequalities in One Variable

Solving a linear inequality:

Solve a linear inequality just like a linear equation, by performing operations to both sides of the inequality in order to isolate the variable. The only difference is that when dividing or multiplying both sides of the inequality by a negative number, the direction of the inequality changes (i.e. if starting with $<$ or \leq , switch it to $>$ or \geq and visa versa).

Example:

$$\begin{aligned}1 - 5x &\geq 2(5 - x) \\1 - 5x &\geq 10 - 2x \\1 - 3x &\geq 10 \\-3x &\geq 9 \\x &\leq -3 \leftarrow \text{Change direction.}\end{aligned}$$

Writing the Solution:

Set Builder Notation:

Write the solution in braces as seen below.

$\{x|x \leq -3\}$ (Read as: “ x such that x is less than or equal to negative 3”)

Interval Notation:

Write the solution in interval form, using parenthesis () to indicate values that are excluded and square brackets [] to indicate values that are included.

Note: **Always** use parenthesis with ∞ and $-\infty$.

Examples:

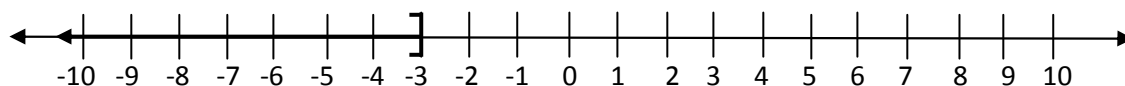
- $x \leq -3$ in interval notation would be $(-\infty, -3]$
- $x > 7$ in interval notation would be $(7, \infty)$
- $-\frac{3}{4} \leq x < \frac{2}{3}$ in interval notation would be $[-\frac{3}{4}, \frac{2}{3})$

Graphing a linear inequality:

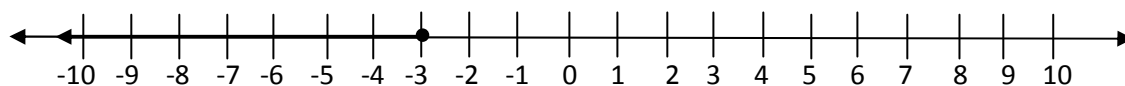
After solving a linear inequality, to graph the solution on a number line, use parenthesis or an open circle to indicate values not included, and use brackets or closed circles to indicate values that are included. Use arrows to indicate that the graph continues toward positive or negative infinity.

Examples:

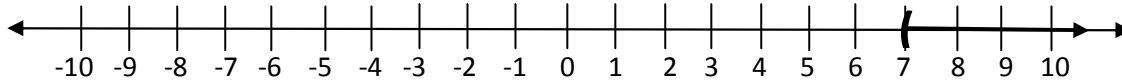
a. $x \leq -3$



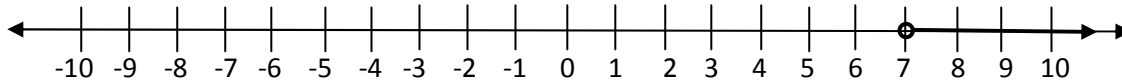
or



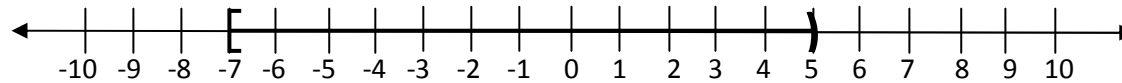
b. $x > 7$



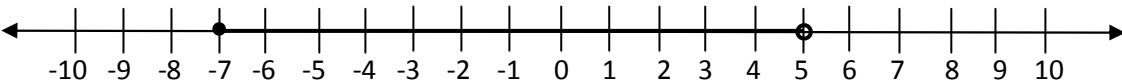
or



c. $-7 \leq x < 5$



or



Solving Linear Absolute Value Equations and Inequalities:

Solving Linear Absolute Value Equations:

1. Isolate the absolute value.
2. If absolute value equals zero, remove absolute value symbols & solve equation to get **one solution**.
3. If absolute value equals a negative number, there is **no solution**.
4. If absolute value equals a number greater than zero, set the argument (expression within the absolute value) equal to the number **and** set it equal to the opposite of the number, using an 'or' statement in between the two equations. Then solve each equation to get **two solutions**.

Examples:

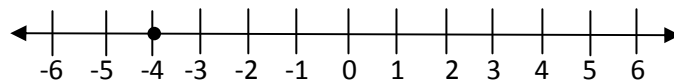
a. $|3x + 12| + 7 = 7$

$|3x + 12| = 0$ ← Isolate Absolute Value

$3x + 12 = 0$

$3x = -12$

$x = -4$



b. $|3x - 7| + 7 = 2$

$|3x - 7| = -5$

No Solution

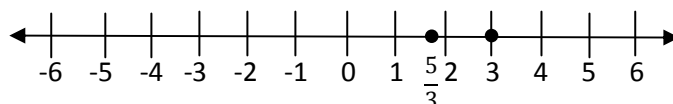
c. $|3x - 7| + 7 = 9$

$|3x - 7| = 2$

$3x - 7 = 2$ or $3x - 7 = -2$

$3x = 9$ or $3x = 5$

$x = 3$ or $x = \frac{5}{3}$



Solving Linear Absolute Value Inequalities:

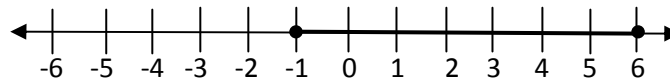
1. Isolate the absolute value.
2. If absolute value is less than zero, there is **no solution**.
3. If absolute value is less than or equal to zero, there is **one solution**. Just set argument equal to zero and solve.
4. If absolute value is greater than or equal to zero, the solution is **all real numbers**.
5. If absolute value is greater than zero, the solution is all real numbers **except** the value which makes it equal to zero.
6. If absolute value is less than or less than or equal to a positive number, place the argument in a 3-part inequality between the opposite of the number and the number, then solve.
7. If absolute value is greater than or greater than or equal to a positive number, set the argument less than the opposite of the number **and** greater than the number using an 'or' statement in between the two inequalities. Then solve each inequality, writing the solution as a union of the two solutions.

Examples:

- a. $|2x - 5| \leq 7$
 $-7 \leq 2x - 5 \leq 7$ Note: To solve a 3-part inequality, perform the same operations
 $-2 \leq 2x \leq 12$ to all three parts in order to isolate the variable in the middle.
 $-1 \leq x \leq 6$

Interval Notation: $[-1, 6]$

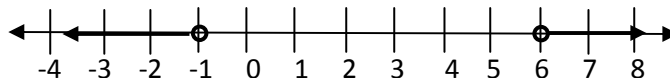
Graph:



- b. $|2x - 5| > 7$
 $2x - 5 < -7$ or $2x - 5 > 7$
 $2x < -2$ or $2x > 12$
 $x < -1$ or $x > 6$

Interval Notation: $(-\infty, -1) \cup (6, \infty)$

Graph:



Solving Polynomial and Rational Inequalities:

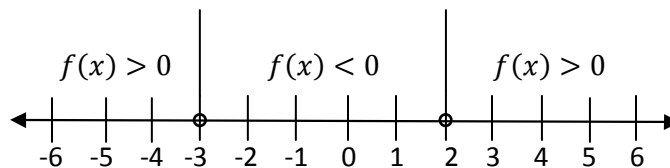
Solving Polynomial Inequalities:

Given $f(x)$ is a polynomial in standard form.

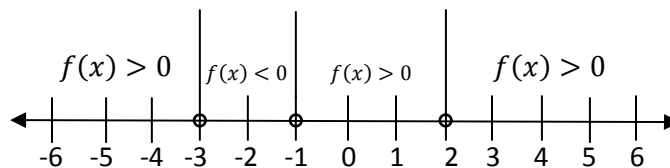
1. Write f in completely factored form.
2. Plot real zeros on the number line.
3. Choose test points in each interval to the left, to the right, and in between the zeros. Plug test points into the factored form of the polynomial to determine if that interval is positive (greater than zero) or negative (less than zero).
4. Write the solution in interval notation.

Examples:

- a. $f(x) = x^2 + x - 6$, solve $f(x) > 0$.
 $f(x) = (x + 3)(x - 2)$, zeros: -3 and 2
 Test Points: $-4, 0$, and 3
 $(-4 + 3)(-4 - 2) = (-1)(-6) = 6 > 0$
 $(0 + 3)(0 - 2) = (3)(-2) = -6 < 0$
 $(3 + 3)(3 - 2) = (6)(1) = 6 > 0$
 Solution: $(-\infty, -3) \cup (2, \infty)$



- b. $f(x) = (x + 1)(x - 2)^2(x + 3)$, solve $f(x) \leq 0$.
 Zeros: $-3, -1$ and 2
 Test Points: $-4, -2, 0$, and 3
 $(-4 + 1)(-4 - 2)^2(-4 + 3) = 108 > 0$
 $(-2 + 1)(-2 - 2)^2(-2 + 3) = -16 < 0$
 $(0 + 1)(0 - 2)^2(0 + 3) = 12 > 0$
 $(3 + 1)(3 - 2)^2(3 + 3) = 24 > 0$
 Solution: $[-3, -1]$



Solving Rational Inequalities:

1. Set rational inequality less than, less than or equal to, greater than, or greater than or equal to zero and write as **one** fraction.
2. Completely factor both the numerator and the denominator.
3. Plot the zeros and the values which make the rational expression undefined (asymptotes) on the number line.
4. Choose test points in each interval to the left, to the right, and in between the zeros and asymptotes. Plug test points into the factored form of the rational expression to determine if that interval is positive (greater than zero) or negative (less than zero).
5. Write the solution in interval notation.

Example:

Solve:

$$\frac{x^2 - 9}{x^3 - x^2 - x + 1} \leq 0$$

$$\frac{(x-3)(x+3)}{(x-1)^2(x+1)} \leq 0 \quad \text{Zeros: } -3 \text{ and } 3, \text{ Asymptotes: } 1 \text{ and } -1, \text{ \& Test Points: } -4, -2, 0, 2, \text{ and } 4$$

Test the Intervals Using Test Points

Since the rational expression is ≤ 0 , the intervals for which the expression is negative need to be found.

$$\frac{(-4-3)(-4+3)}{(-4-1)^2(-4+1)} = \frac{+}{-} \text{ is negative.} \quad \frac{(0-3)(0+3)}{(0-1)^2(0+1)} = \frac{-}{+} \text{ is negative.} \quad \frac{(4-3)(4+3)}{(4-1)^2(4+1)} = \frac{+}{+} \text{ is positive.}$$

$$\frac{(-2-3)(-2+3)}{(-2-1)^2(-2+1)} = \frac{-}{-} \text{ is positive.} \quad \frac{(2-3)(2+3)}{(2-1)^2(2+1)} = \frac{-}{+} \text{ is negative.}$$

Solution: $(-\infty, -3] \cup (-1, 1) \cup (1, 3]$

