Rational Expressions

A quotient of two integers, $\frac{a}{b}$, where $b \neq 0$, is called a **rational expression**.

Some examples of rational expressions are $\frac{7x}{9}, \frac{12}{x+4}, \frac{3x+1}{2x-5}$, and $\frac{x^2-10}{x^3-x^2+3}$. When $x = -4$, the denominator of the expression $\frac{12}{x+4}$ becomes 0 and the expression is meaningless. Mathematicians state this fact by saying that the expression $\frac{12}{x+4}$ is undefined when $x = -4$. One can see that the value $x = \frac{5}{2}$, makes the expression $\frac{3x+1}{2x-5}$ undefined. On the other hand, when any real number is substituted into the expression $\frac{7x}{9}$, the answer is always a real number. There are no values for which this expression is undefined.

EXAMPLE Determine the value or values of the variable for which the rational expression is defined.

a) $\frac{x+3}{2x-5}$ b) $\frac{x+3}{x^2+6x-7}$

Solution a) Determine the value or values of *x* that make *2x – 5* equal to 0 and exclude these. This can be done by setting *2x – 5* equal to 0 and solving the equation for *x*.

$$2x - 5 = 0$$

$$2x = 5$$

$$x = \frac{5}{2}$$

Do not consider $x = \frac{5}{2}$ when considering the rational expression $\frac{x+3}{2x-5}$. This expression is defined for all real numbers except $x = \frac{5}{2}$. Sometimes to shorten the answer it is written as $x \neq \frac{5}{2}$.

b) To determine the value or values that are excluded, set the denominator equal to zero and solve the equation for the variable.

$$x^2 + 6x - 7 = 0$$

$$(x + 7)(x - 1) = 0$$

$$x + 7 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = -7 \qquad\qquad x = 1$$

Therefore, do not consider the values $x = -7$ or $x = 1$ when considering the rational expression $\frac{x+3}{x^2+6x-7}$. Both $x = -7$ and $x = 1$make the denominator zero. This is defined for all real numbers except $x = -7$ and $x = 1$. Thus, $x \neq -7$ and $x \neq 1$.

SIGNS OF A FRACTION

$\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}$ Notice: $-\frac{a}{b} \neq \frac{-a}{-b}$

Generally, a fraction is not written with a negative denominator. For example, the expression $\frac{2}{-5}$ would be written as either $\frac{-2}{5}$ or $-\frac{2}{5}$. The expression $\frac{x}{-(4-x)}$ can be written $\frac{x}{x-4}$ since $-(4 - x) = -4 + x$ or $x - 4$.

Other examples of equivalent fractions:

$$-\frac{y}{y-2}=\frac{-y}{y-2}=\frac{y}{2-y} \qquad -\frac{x-2}{x-3}=\frac{2-x}{x-3}=\frac{x-2}{3-x} \qquad \frac{z-5}{7-z}=-\frac{5-z}{7-z}=\frac{5-z}{z-7}$$

SIMPLIFYING RATIONAL EXPRESSIONS

A rational expression is **simplified** or **reduced to its lowest** terms when the numerator and denominator have no common factors other than 1. The fraction $\frac{9}{12}$ is not simplified because 9 and 12 both contain the common factor 3. When the 3 is factored out, the simplified fraction is $\frac{3}{4}$.

$$\frac{9}{12}=\frac{{}^{1}\cancel{3}\cdot 3}{{}^{1}\cancel{3}\cdot 4}=\frac{3}{4}$$

The rational expression $\frac{ab-b^2}{2b}$ is not simplified because both the numerator and denominator have a common factor, *b*. to simplify this expression, factor *b* from each term in the numerator, then divide it out.

$$\frac{ab-b^2}{2b}=\frac{\cancel{b}(a-b)}{2\cancel{b}}$$

$$=\frac{a-b}{2}$$

Thus, $\frac{ab-b^2}{2b}$ becomes $\frac{a-b}{2}$ when simplified.

To Simplify Rational Expressions
1. Factor both the numerator and denominator as completely as possible. 2. Divide out any factors common to both the numerator and denominator.

Example 1 Simplify $\frac{5x^3+10x^2-25x}{10x^2}$

Solution Factor the greatest common factor, $5x$, from each term in the numerator. Since $5x$ is a factor common to both the numerator and denominator, divide it out.

$$\frac{5x^3+10x^2-25x}{10x^2}=\frac{\cancel{5x}(x^2+2x-5)}{\cancel{5x}\cdot 2x}$$

$$=\frac{x^2+2x-5}{2x}$$

Example 2 Simplify $\frac{x^2+2x-3}{x+3}$

Solution Factor the numerator; then divide out the common factor.

$$\frac{x^2+2x-3}{x+3}=\frac{\cancel{(x+3)}(x-1)}{\cancel{x+3}}$$

$$=x-1$$

Example 3 Simplify $\frac{x^2-16}{x-4}$

Solution Factor the numerator; then divide out common factors.

$$\frac{x^2-16}{x-4} = \frac{(x+4)\cancel{(x-4)}}{\cancel{x-4}}$$

$$= x - 4$$

Example 4 Simplify $\frac{2x^2+7x+6}{x^2-x-6}$

Solution Factor both the numerator and denominator, then divide out common factors.

$$\frac{2x^2+7x+6}{x^2-x-6} = \frac{(2x+3)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}}$$

$$= \frac{2x+3}{x-3}$$

Example 5 Simplify $\frac{2m^2+4m-6}{4m^2+16m+12}$

Solution Factor both the numerator and denominator, then divide out common factors.

$$\frac{2m^2+4m-6}{4m^2+16m+12} = \frac{\cancel{2}\cancel{(m+3)}(m-1)}{{}_2\cancel{4}\cancel{(m+3)}(m+1)}$$

$$= \frac{m-1}{2(m+1)}$$

Example 6 Simplify $\frac{x^3-1}{x^2-2x+1}$

Solution Factor both the numerator and denominator, then divide out common factors.

$$\frac{x^3-1}{x^2-2x+1} = \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x-1)}$$

$$= \frac{x^2+x+1}{x-1}$$

Example 7 Simplify $\frac{2x+4-x^3-2x^2}{x^2+5x+6}$

Solution Factor both numerator and denominator, then divide out common factors.

$$\frac{2x+4-x^3-2x^2}{x^2+5x+6} = \frac{2(x+2)-x^2(x+2)}{(x+2)(x+3)}$$

$$= \frac{\cancel{(x+2)}(2-x^2)}{\cancel{(x+2)}(x+3)}$$

$$= \frac{2-x^2}{x+3}$$

Consider the expression $\frac{5x-3}{x+3}$, a common student error is to attempt to cancel the *x* or the 3 or both *x* and 3 appearing in this expression.

This is **WRONG!** $\frac{5\ {}_1\cancel{x}-\ {}_1\cancel{3}}{{}_1\cancel{x}+\ {}_1\cancel{3}}$ does **not** equal $\frac{5-1}{1+1}=\frac{4}{2}=2$

It is **WRONG** because factors are not being reduced. Evaluating this expression for an easy value, such as $x=1$, would show that the illustrated cancellations are **WRONG**. If $x=1$, $\frac{5x-3}{x+3}$ becomes $\frac{5(1)-3}{1+3}=\frac{2}{4}=\frac{1}{2}$.

Remember: Only common *factors* can be divided out from expressions.

$$\frac{20x^2}{4x}=5x \qquad\qquad \frac{x^2-20}{x-4}$$

In the denominator of the example on the left, $4x$, the 4 and x are factors since they are *multiplied* together. The 4 and the x are also both factors of the numerator $20x^2$, since $20x^2$ can be written $4\cdot x\cdot 5\cdot x$.

Some students incorrectly divide out terms. In the expression $\frac{x^2-20}{x-4}$, the x and –4 are terms of the denominator, not factors, and therefore cannot be divided out.

Recall that when -1 is factored from a polynomial, the sign of each term in the polynomial changes.

EXAMPLES: $-3x+5=-1(3x-5)=-(3x-5)$

$6-2x=-1(-6+2x)=-(2x-6)$

Example 8 Simplify $\frac{3x-7}{7-3x}$

Solution Since each term in the numerator differs only in sign from its like term in the denominator, factor-1 from each term in the denominator.

$$\frac{3x-7}{7-3x}=\frac{3x-7}{-1(-7+3x)}$$

$$=\frac{3x-7}{-(3x-7)}$$

$$=-1$$

Example 9 Simplify

Solution $\frac{4x^2-23x-6}{6-x}=\frac{(4x+1)(x-6)}{6-x}$

$$\frac{(4x+1)(x-6)}{6-x}=\frac{(4x+1)\cancel{(x-6)}}{-1\cancel{(x-6)}}$$

$$=\frac{4x+1}{-1}$$

$$=-(4x+1)$$

ADDITIONAL EXERCISES

Determine the value or values of the variables for which the expression is defined.

1. $\frac{x+9}{x^2-x-12}$
2. $\frac{x-3}{5x-3}$
3. $\frac{x^2+5x-36}{x^2+7x+6}$
4. $\frac{64x^2-25}{-8x^2-10x}$
5. $\frac{x^2+5x-36}{x^2-8x+12}$
6. $\frac{16x^2-9}{-9x^2-10x}$

Simplify

7. $\frac{9f+fg}{4f}$
8. $\frac{5p+pq}{8p}$
9. $\frac{4f+fg}{7f}$
10. $\frac{12x-24}{8-4x}$
11. $\frac{6x-24}{12-3x}$
12. $\frac{x^2+2x-35}{5-x}$
13. $\frac{x^2-5x-14}{7-x}$
14. $\frac{x^2-5x-14}{x^2-49}$
15. $\frac{u-1}{u^2-1}$
16. $\frac{x^2-x-12}{x^2-16}$
17. $\frac{j+8}{j^2-64}$
18. $\frac{x^2+5x-14}{2-x}$
19. $\frac{x^2+7x-18}{x^2-4}$
20. $\frac{x^2-9}{27-x^3}$

Answers

1. $x \neq -3, x \neq 4$
2. $x \neq \frac{3}{5}$
3. $x \neq -6, x \neq -1$
4. $x \neq 0, x \neq -\frac{5}{4}$
5. $x \neq 6, x \neq 2$
6. $x \neq -\frac{10}{9}, x \neq 0$
7. $\frac{9+g}{4}$
8. $\frac{5+q}{8}$
9. $\frac{4+g}{7}$
10. -3
11. -2
12. $-(x+7)$
13. $-(x+2)$
14. $\frac{x+2}{x+7}$
15. $\frac{1}{u+1}$
16. $\frac{x+3}{x+4}$
17. $\frac{1}{j-8}$
18. $-(x+7)$
19. $\frac{x+9}{x+2}$
20. $\frac{-(x+3)}{9+3x+x^2}$