Inequalities in One Variable (Linear, Polynomial, & Rational)

Solving a Linear Inequality: (i.e. $ax + b \geq c$)

Solving Inequalities:

Solve a linear inequality just like a linear equation, by performing operations to both sides of the inequality in order to *isolate the variable*. The only difference is that when **dividing or multiplying** both sides of the inequality **by a negative** number, the direction of the inequality **sign changes** (i.e. if starting with $<$ or $\leq$, switch it to $>$ or $\geq$ and vice versa).

Inequalities can also be **Compound Inequalities** meaning there is more than one inequality listed. Compound Inequalities joined by the word **"and"** represent intersections (*where both conditions are met*) and those joined by the word **"or"** represent unions (*where either condition is met*).

Examples:

a.
$$1 - 5x \geq 2(5 - x)$$
$$1 - 5x \geq 10 - 2x$$
$$1 - 3x \geq 10$$
$$-3x \geq 9$$
$$\rightarrow x \leq -3$$
Change the direction of the inequality sign!

b.
$$-4 \leq 2x + 2 < 10$$
$$-6 \leq 2x < 8$$
$$-3 \leq x < 4$$
This is a "Compound" Inequality problem with an Intersection. It could also be written as $-4 \leq 2x + 2$ *and* $2x + 2 < 10$

c.
$$x + 2 \geq 0 \quad or \quad x - 4 < 0$$
$$x \geq -2 \quad or \quad x < 4$$
Therefore x can be any real number ($x \in \mathbb{R}$)

This is a "Compound" Inequality problem with a Union.

Writing the Solution for Inequalities:

Solutions can be written in either **Set Builder Notation** or **Interval Notation**.

Set Builder Notation:
Write the solution in braces { }.

Example:

a. $x \leq -3$ would be written as shown below…

$\{x | x \leq -3\}$ (Read as: "x such that x is less than or equal to negative 3")

Interval Notation:
Write the solution in interval form, using parenthesis ( ) to indicate values that are excluded and square brackets [ ] to indicate values that are included. ***Always** use parenthesis with* $-\infty$ *and* ∞.

Examples:

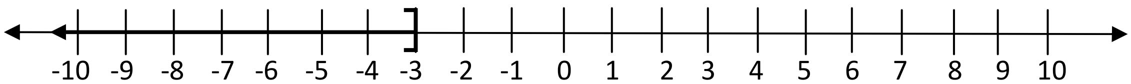
a. $x \leq -3$ in interval notation would be $(-\infty, -3]$
b. $x > 7$ in interval notation would be $(7, \infty)$
c. $-\frac{3}{4} \leq x < \frac{2}{3}$ in interval notation would be $\left[-\frac{3}{4}, \frac{2}{3}\right)$ *(intersection)*
d. $x > 4$ ***and*** $x \leq 11$ in interval notation would be $(4, 11]$ *(intersection)*
e. $x \geq 10$ ***and*** $x < 3$ would have No Solution *(intersection)*
e. $x \leq -2$ ***or*** $x > 1$ in interval notation would be $(-\infty, -2] \cup (1, \infty)$ *(union)*
f. $x < 6$ ***or*** $x \geq 1$ in interval notation would be $(-\infty, \infty)$ or all Real #'s ($\mathbb{R}$) *(union)*

Graphing a Linear Inequality:

After solving a linear inequality, to graph the solution on a number line, use **parenthesis or an open circle** to indicate **values <u>not</u> included**, and use **brackets or closed circles** to indicate **values that <u>are</u> included**. Use arrows to indicate that the graph continues toward positive or negative infinity.

Examples:

a. $x \leq -3$ (Brackets or Closed Circle)



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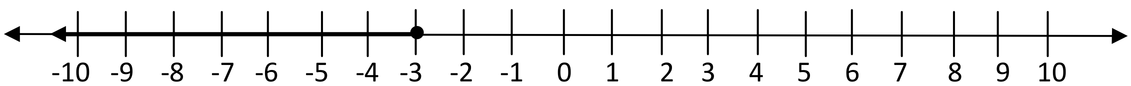
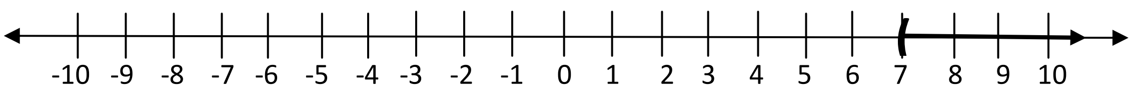


b. $x > 7$ (Parenthesis or Open Circle)



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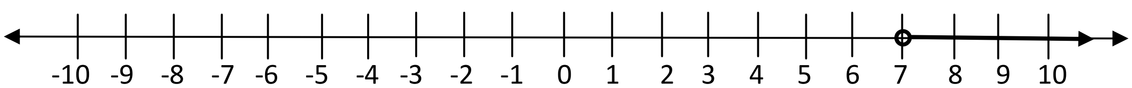
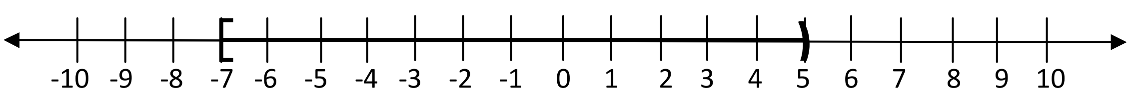


c. $-7 \leq x < 5$ (Combination – both *included* and *not included* values)



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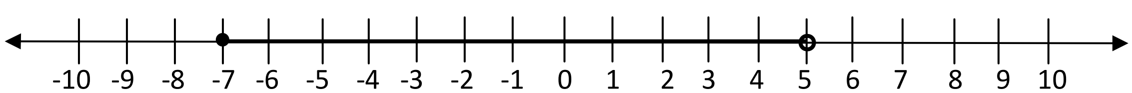


Solving Polynomial Inequalities: (i.e. $ax^2 + bx + c \geq 0$)

Solving:

1. Set the polynomial inequality less than, less than or equal to, greater than, or greater than or equal to zero.
2. Completely factor the polynomial.
3. Solve for the zeros.
4. Plot the *real* zeros on the number line. *(Do not plot any zeros that are imaginary.)*
5. Choose **test points** in each interval to the left, to the right, and in between the zeros. Plug test points into the factored form of the polynomial to determine if that interval is positive (greater than zero) or negative (less than zero). *(Note: it is recommended to choose integers when possible.)*
6. Write the solution in interval notation, paying close attention to whether parenthesis or brackets are required.

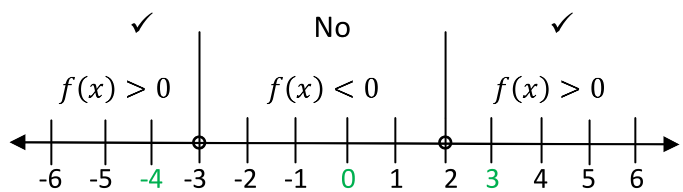
Examples:

a. $x^2 + x > 6$

Because the inequality sign is only greater than (>), open circles will be used on # line

Step 1. $x^2 + x - 6 > 0$
Step 2. $(x+3)(x-2) > 0$
Step 3. $x + 3 = 0$ and $x - 2 = 0.$ So, $x = -3$ and $x = 2$ are the zeros.
Step 4. Plot $x = -3$ and $x = 2$ on the number line. Use *open* circles.
Step 5. Choose test points: -4, 0, and 3
Plug in Test Points to determine which intervals work *(i.e. which ones are > 0)*.

-4: $(-4+3)(-4-2) = (-1)(-6) = \mathbf{6 > 0}$
0: $(0+3)(0-2) = (3)(-2) = -6 < 0$
3: $(3+3)(3-2) = (6)(1) = \mathbf{6 > 0}$

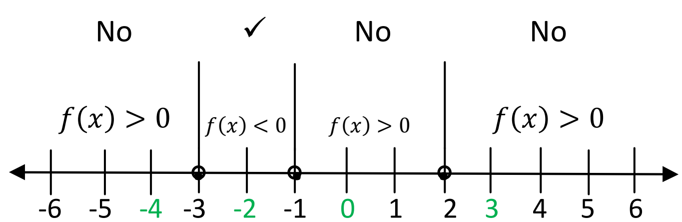


Step 6. Solution: $(-\infty, -3) \cup (2, \infty)$

b. $f(x) = (x+1)(x-2)^2(x+3)$, solve $f(x) \leq 0$.

Because the inequality sign is less than **or equal to** ($\leq$), closed circles will be used on # line

Step 1. $(x+1)(x-2)^2(x+3) \leq 0$
Step 2. Already factored
Step 3. Zeros: $-3, -1$ and 2
Step 4. Plot $x = -3, -1,$ and 2 on the number line. Use *closed* circles.
Step 5. Choose test points: $-4, -2, 0,$ and 3
Plug in Test Points to determine which intervals work *(i.e. which ones are ≤ 0).*
-4: $(-4+1)(-4-2)^2(-4+3) = 108 > 0$
-2: $(-2+1)(-2-2)^2(-2+3) = \mathbf{-16 \leq 0}$
0: $(0+1)(0-2)^2(0+3) = 12 > 0$
3: $(3+1)(3-2)^2(3+3) = 24 > 0$



Step 6. Solution: $[-3, -1]$

Solving Rational Inequalities: (i.e. $\frac{ax^2+bx+c}{dx^2+ex+f} \geq 0$)

1. Set the rational inequality less than, less than or equal to, greater than, or greater than or equal to zero and write as **one** fraction (*find LCD if needed*).
2. Completely factor both the numerator and the denominator.
3. Identify the **Critical Numbers** - There are 2 types of critical numbers: *1) Zeros from the numerator and 2) Values from the denominator which make the rational expression undefined.*
4. Plot the Critical Numbers on the number line. The **zeroes** can be either **open or closed** circles depending on the type of inequality sign. The "**undefined**" values from the denominator will **always be open circles**.
5. Choose test points in each interval to the left, to the right, and in between the Critical Numbers. Plug test points into the factored form of the rational expression to determine if that interval is positive (greater than zero) or negative (less than zero).
6. Write the solution in interval notation, paying close attention to whether the values have parenthesis or brackets.

Example: $\frac{x^2-9}{x^3-x^2-x+1} \leq 0$ ← Because the inequality sign is less than **or equal to** ($\leq$), closed circles will be used for the zeros on the # line.

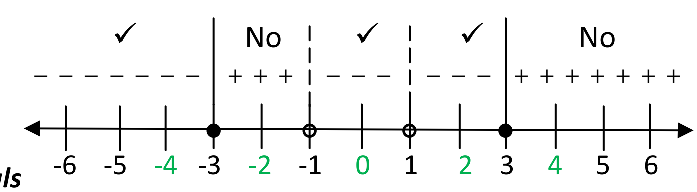
Step 1. $\frac{x^2-9}{x^3-x^2-x+1} \leq 0$ *(Rational inequality was already written as 1 fraction.)*

Step 2. $\frac{(x-3)(x+3)}{(x-1)^2(x+1)} \leq 0$ *(Factor: Num: Diff of Sqrs; Den: Grouping, then Diff of Sqrs)*

Step 3. Critical Numbers: Numerator (Zeros) → $x = 3$ *and* -3
(Set factors = 0 and solve) Denominator (Asymptotes) → $x = 1$ *and* -1

Step 4. Plot Critical Numbers (CN's) on number line.
Use closed circles for the Zeros and open circles for the Asymptotes.

Step 5. Choose Test Points:
-4, -2, 0, 2, and 4



Since this rational expression is ≤0, the intervals for which the expression is negative need to be found

Plug in Test Points to determine which intervals work (*i.e. which ones are* ≤ 0).

(-4) $\frac{(-4-3)(-4+3)}{(-4-1)^2(-4+1)} = \frac{+}{-}$ is Negative ***(-2)*** $\frac{(-2-3)(-2+3)}{(-2-1)^2(-2+1)} = \frac{-}{-}$ is Positive

(0) $\frac{(0-3)(0+3)}{(0-1)^2(0+1)} = \frac{-}{+}$ is Negative ***(2)*** $\frac{(2-3)(2+3)}{(2-1)^2(2+1)} = \frac{-}{+}$ is Negative

(4) $\frac{(4-3)(4+3)}{(4-1)^2(4+1)} = \frac{+}{+}$ is Positive

Step 6. Solution: $(-\infty, -3] \cup (-1, 1) \cup (1, 3]$

Inequalities Practice Problems

	Problem	**Answer**	**Type**
1.	$x+9 < 21$	$\{x \mid x < 12\}$ OR $(-\infty, 12)$	Linear
2.	$-2x+4 > -12$	$\{x \mid x < 8\}$ OR $(-\infty, 8)$	Linear
3.	$4-3x \leq 2(7-x)$	$\{x \mid x \geq -10\}$ OR $[-10, \infty)$	Linear
4.	$-3x+6<9$ or $4x+2 \leq -10$	$(-\infty, -3] \cup (-1, \infty)$	Linear - Compound
5.	$\frac{3}{4} > 2x-1 > \frac{1}{4}$	$\left\{x \middle\vert \frac{5}{8} < x < \frac{7}{8}\right\}$ OR $\left(\frac{5}{8}, \frac{7}{8}\right)$	Linear - Compound
6.	$2x+4 < 6$ and $-3x+10 < 1$	No Solution	Linear - Compound
7.	$5x+25 > 0$ or $-2x+10 > 4$	$\mathbb{R}$ OR $(-\infty, \infty)$	Linear - Compound
8.	$x^2+4 > 0$	$\mathbb{R}$ OR $(-\infty, \infty)$	Polynomial
9.	$x^2+3x+2 < 0$	$\{x \mid -2 < x < -1\}$ OR $(-2, -1)$	Polynomial
10.	$x^2 \leq 16$	$\{x \mid -4 \leq x \leq 4\}$ OR $[-4, 4]$	Polynomial
11.	$x^2-x > 20$	$(-\infty, -4) \cup (5, \infty)$	Polynomial
12.	$x^2+2x+1 \leq 0$	$\{-1\}$	Polynomial
13.	$x^3+2x^2-4x \leq 8$	$(-\infty, 2]$	Polynomial
14.	$x^2-6x > -9$	$(-\infty, 3) \cup (3, \infty)$	Polynomial
15.	$x^2+25 \leq 0$	No Real Solution	Polynomial
16.	$\frac{t^2-1}{t} < 0$	$(-\infty, -1) \cup (0, 1)$	Rational
17.	$\frac{x^2+4x}{x^2-4} \leq 0$	$[-4, -2) \cup [0, 2)$	Rational
18.	$\frac{m+4}{m-2} - 4 \geq 0$	$(2, 4]$	Rational
19.	$-\frac{7p}{p^2-100} \geq \frac{p+2}{p+10}$	$(-10, -4] \cup [5, 10)$	Rational
20.	$\frac{1}{n-3} - \frac{1}{n+3} \leq 2$	$(-\infty, -2\sqrt{3}] \cup (-3, 3) \cup [2\sqrt{3}, \infty)$	Rational