Linear Asymptotes and Holes

Graphs of Rational Functions can contain linear asymptotes. These asymptotes can be Vertical, Horizontal, or Slant (*also called Oblique*). Graphs may have more than one type of asymptote. Given a Rational Function $f(x)$, the steps below outline how to find the asymptote(s).

n = Degree of Numerator

Rational Function = $f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$

m = Degree of Denominator

<u>Vertical Asymptotes</u> (VA) – The line $\boldsymbol{x = c}$ is a Vertical Asymptote of the graph of a rational function *when* $f(x) \to \pm\infty$, as $x \to c$ from the right or the left. *They are graphed as dashed vertical lines.*

The graph of $f(x)$ has Vertical Asymptotes at the real zeros of $D(x)$. *Find the VA's by **setting the denominator of the <u>simplified</u> function equal to "0" and solving the resulting equation**. Note: Verify that the real zeros of $D(x)$ are not actually "holes" first by factoring $N(x)$ and $D(x)$ and simplifying the function if possible. (See "Holes*" at bottom of second page.)*

<u>Horizontal Asymptotes</u> (HA) – The line $\boldsymbol{y = c}$ is a Horizontal Asymptote of the graph of a rational function *when* $f(x) \to c$, as $x \to \pm\infty$. *They are graphed as dashed horizontal lines.*

The graph of $f(x)$ either has 1 Horizontal Asymptote or no HA which is *determined by comparing the degree (n) of the Numerator $N(x)$ with the degree (m) of the Denominator $D(x)$.*
There are 3 cases to consider.

<u>Case 1:</u>

If $n < m$

HA: $y = 0$

<u>Case 2:</u>

If $n = m$

HA: $y = \frac{a_n}{b_m} = \frac{\text{leading coef num}}{\text{leading coef den}}$

<u>Case 3:</u>

If $n > m$

HA: None → But there ***could be*** a Slant Asymptote!

Examples: Find all Vertical and Horizontal Asymptotes of the graphs of the Rational Functions.

a) $f(x) = \frac{x-2}{x^2-9} = \frac{x-2}{(x+3)(x-3)}$

Set $(x+3)(x-3) = 0$

So $x = -3,\ 3$

VA: $\boldsymbol{x = -3,\ x = 3}$

HA: $n < m$, so $\boldsymbol{y = 0}$

b) $f(x) = \frac{3x^2-12}{x^2+2x-3} = \frac{3(x+2)(x-2)}{(x+3)(x-1)}$

Set $(x+3)(x-1) = 0$

So $x = -3,\ 1$

VA: $\boldsymbol{x = -3,\ x = 1}$

HA: $n = m$, so $\boldsymbol{y} = \frac{a_m}{b_m} = \frac{3}{1}$ = **3**

Slant *(or Oblique)* **Asymptotes (SA)** – The line $\boldsymbol{y = mx + b}$ is a Slant Asymptote of the graph of a rational function *if* as $x \to \pm\infty$, $f(x) \to$ *the line* $y = mx + b$. *They are graphed as dashed lines.*

If the degree of the numerator *(n)* is **exactly 1 more** than the degree of the denominator *(m)*, then there **could be** a Slant Asymptote.

To find the Slant Asymptote:

1. Factor the numerator and denominator of $f(x)$.
2. Identify and "reduce" any holes*. *(See next section.)*
3. Multiply the numerator and denominator back in to polynomials if necessary and then divide the remaining numerator by the remaining denominator (i.e. use long division).
4. **Set the resulting quotient equal to y**. Ignore the remainder.
5. The resulting equation, $y = mx + b$, is the Slant Asymptote.

Examples: Find any VA's and Slant Asymptotes of the graphs of the Rational Functions.

a) $f(x) = \frac{x^2-3x-4}{x-2} = \frac{(x-4)(x+1)}{x-2}$ ← *No Hole

Long Division: (Quotient →)

$$x - 2\,\overline{)\,x^2 - 3x - 4}$$

Quotient: $x - 1$

$$\begin{array}{r} x^2 - 2x \\ \hline -x - 4 \\ -x + 2 \\ \hline -6 \end{array}$$

SA: $\boldsymbol{y = x - 1}$
HA: None
VA: $\boldsymbol{x = 2}$

b) $f(x) = \frac{x^2+x-6}{x+3} = \frac{(x+3)(x-2)}{x+3}$ ← *Hole

So, $f(x) = x - 2$
with a hole at $x = -3$

SA: None
(function no longer has a denominator)
HA, VA: None

Holes* - Sometimes, graphs of Rational Functions can contain a "Hole(s)". This occurs when a common (real) factor shows up in the numerator and denominator. This value of x is still a domain restriction, but it is represented as a "Hole" in the graph of $f(x)$ vs. as a Vertical Asymptote.

Example - Find any Horizontal, Vertical, or Slant Asymptotes of $f(x)$. Also identify any Holes.

$$f(x) = \frac{2x^3-x^2-2x+1}{x^2+3x+2} = \frac{(x+1)(x-1)(2x-1)}{(x+1)(x+2)} = \frac{(x-1)(2x-1)}{(x+2)} = \frac{2x^2-3x+1}{x+2} \quad n > m \textit{ by exactly } 1$$

Hole at $x = -1$ | Use to find VA | Perform long division to find the SA

Holes: @ $x = -1$

VA: $x = -2$

HA: None $(n > m)$

SA: $y = 2x - 7$